Smooth spaces of actions on the line

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Notation

G a countable group.

Context: Group actions on the line

Let $G \overset{\varphi}{\curvearrowright} \mathbb{R}$ by orientation-preserving homeomorphisms, assumed to be *irreducible* (i.e. no global fixed points).

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Examples: free groups, surface groups, torsion-free nilpotent groups, wreath products.

Non-examples: torsion groups, $\mathrm{SL}_3(\mathbb{Z})$.

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Example: $\mathrm{BS}(2,1) = \langle g, t \mid gtg^{-1} = t^2 \rangle \cong \mathbb{Z}[1/2] \rtimes \mathbb{Z}$ acts faithfully by $g \colon x \mapsto 2x$, $t \colon x \mapsto x+1$.

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Example: $\mathbb{Z}^2 = \langle a, b \rangle$ acts faithfully by $a\colon x \mapsto x + \alpha$, $b\colon x \mapsto x + \beta$ where α, β are rationally independent and $\alpha^2 + \beta^2 = 1$.

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Proposition (Hölder, Rivas)

Up to conjugacy by $\mathrm{Homeo}(\mathbb{R})$, these are the only minimal actions of $\mathbb{Z}^2$ and of $\mathrm{BS}(2,1)$ on $\mathbb{R}$.

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Define a minimal action φ_ω of $G = F_2$ by

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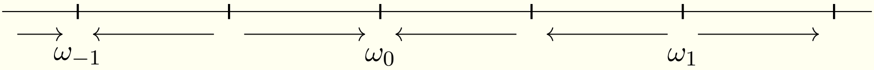

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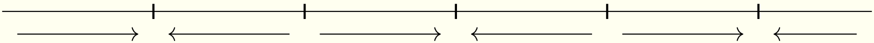

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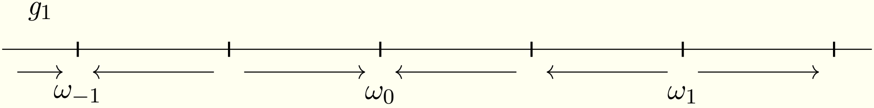

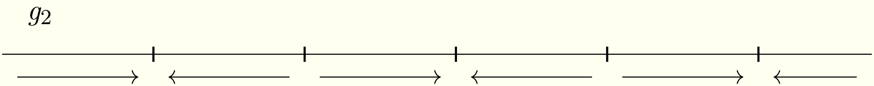

Then φ_ω and $\varphi_{\omega'}$ are conjugate iff ω, ω' are equal up to a power of a shift.

Question, bis: what is the Borel complexity of conjugacy in $\mathrm{Rep}_{\mathrm{min}}(G) := \{\text{minimal actions}\}$? Is the relation even Borel?

Context: Borel complexity

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Example: if H is a locally compact Polish group acting on X, then $E_{H \curvearrowright X} = \{(x, h.x) \mid x \in X, h \in H\}$ is Borel.

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E is *essentially hyperfinite* if $E \sim_B G$ where $G = \nearrow \bigcup_{n \geq 0} G_n$ and each G_n-class is finite.

Proposition (Brum-Matte Bon-Rivas-Triestino, after Deroin-Navas-Kleptsyn-Parwani)

If G is finitely generated then $E_{\mathrm{Homeo}(\mathbb{R}) \curvearrowright \mathrm{Rep}_{\min}(G)}$ is essentially hyperfinite.

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Proposition (Brum-G.-Matte Bon)

$E_{\mathrm{Homeo}(\mathbb{R}) \curvearrowright \mathrm{Rep}_{\min}(G)}$ is always essentially countable.

Definition

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$\mathcal{C}$ contains countable abelian groups and f.g. groups of sub-exponential growth (Hölder, Plante). Question: does it contain all f.g. (elementary) amenable groups?

Results

Theorem A (B-G-MB)

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If $G_n \in \mathcal{C}$, $n \in \mathbb{N}$ are f.g., then $\bigoplus_{n\in\mathbb{N}} G_n \in \mathcal{C}$.

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- all f.g. solvable groups, and
- the wreath product $H \wr G$ where $G, H \in \mathcal{C}$ are f.g.

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$$\mathrm{PAff}_+(\mathbb{R}) = \{g \in \mathrm{Homeo}_0(\mathbb{R}) \mid \text{there is a finite } \mathcal{B} \subset \mathbb{R} \text{ s.t. } g \text{ is affine on each c.c. of } \mathbb{R} \setminus \mathcal{B}\}$$

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Warning: not all actions of $G \leq \mathrm{PAff}_+(\mathbb{R})$ on $\mathbb{R}$ are by piecewise affine homeomorphisms.

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Here, $F =$ piecewise affine dyadic homeomorphisms of $[0, 1]$ ($g \in F$ if it is piecewise affine, locally of the form $x \mapsto 2^k x + b$ where $k \in \mathbb{Z}$, $b \in \mathbb{Z}[1/2]$).

Proofs: more context

Definitions

Let $\varphi \in \mathrm{Rep}_{\min}(G)$.

$x, y \in \mathbb{R}$ are *proximal for* φ if there is $(g_n)_{n\geq 0} \subseteq G$ s.t. $\varphi(g_n).x$, $\varphi(g_n).y$ converge to the same point.

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- φ is *locally proximal* if every point is contained in an open interval whose endpoints are proximal for φ.

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(III) φ is proximal.

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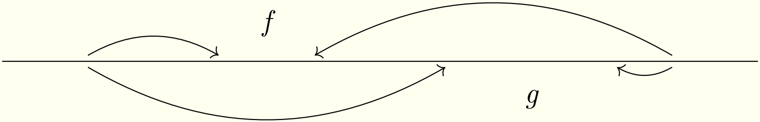

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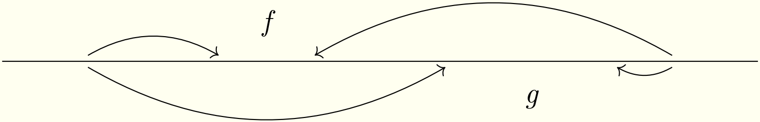

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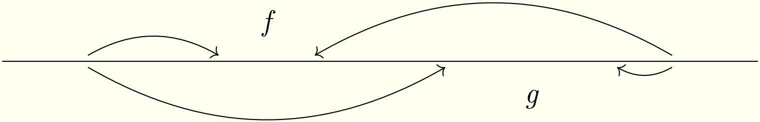

If φ is of type (II), then φ commutes with $f \in \text{Homeo}(\mathbb{R})$ with no fixed points such that the action on $\mathbb{R}/\{x \sim f(x)\} = S^1$ is proximal. Thus G contains non-abelian free groups.

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$G \in \mathcal{C}$ iff for every minimal φ of type III and $(f_n)_{n\geq 0} \subset \mathrm{Homeo}_0(\mathbb{R})$ such that $\lim_n f_n.\varphi = \varphi$, we have $\lim_n f_n = \mathrm{id}_{\mathbb{R}}$

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In particular the orbit equivalence relation on actions of type I or II is always smooth.

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- there is a discrete orbit, or
- there is no minimal subset.

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If $\varphi\big|_N$ is of type III then $\lim_n f_n = \mathrm{id}_{\mathbb{R}}$.

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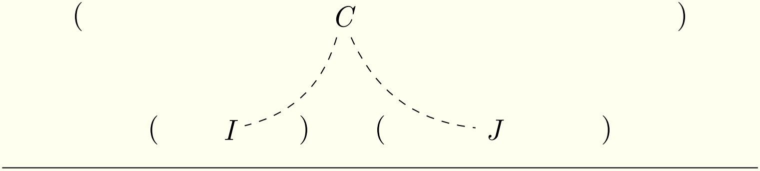

Even more context

Denote by $\mathbb{R} \curvearrowright^{\Psi} \mathrm{Rep}_{\min}(G)$ the *translation flow* conjugating actions by translations: $\Psi_t.\varphi = T_t \circ \varphi \circ T_{-t}$ for $t \in \mathbb{R}$.

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Here, two actions are *pointed conjugate* if they are conjugate via a homeo fixing 0.

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$$\{\text{type II actions}\} \longleftrightarrow \{\text{periodic } \Psi\text{-orbits}\}$$
$$\{\text{type I actions}\} \longleftrightarrow \{\Psi\text{-fixed points}\}$$
$$\{\text{type III actions}\} \longleftrightarrow \{\text{free } \Psi\text{-orbits}\}$$

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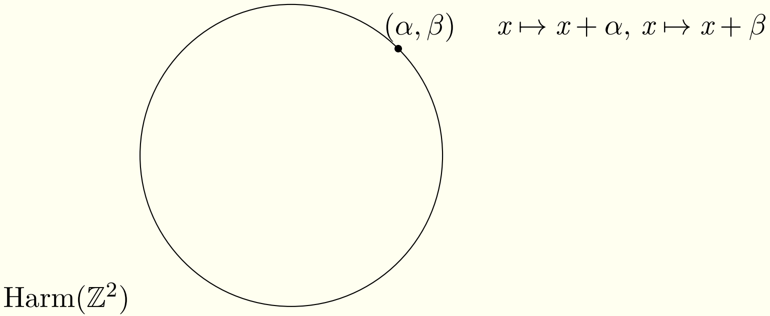
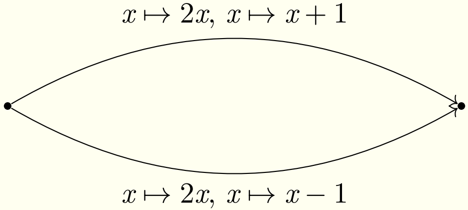
so $\{$conjugacy classes of minimal actions of $G\} = X/\Psi$



(α, β)
$x \mapsto x + \alpha,\ x \mapsto x + \beta$
$\mathrm{Harm}(\mathbb{Z}^2)$

$\mathrm{Harm}(\mathrm{BS}(2,1))$



$\mathrm{Harm}(\mathrm{BS}(2,3))$